

Star products and branes in Poisson-Sigma models

Iván Calvo* and Fernando Falceto†

Departamento de Física Teórica
Universidad de Zaragoza
E-50009 Zaragoza, Spain

E-mail: icalvo@unizar.es, falceto@unizar.es

Abstract

We prove that non-coisotropic branes in the Poisson-Sigma model are allowed at the quantum level. When the brane is defined by second-class constraints, the perturbative quantization of the model yields Kontsevich's star product associated to the Dirac bracket on the brane. Finally, we present the quantization for a general brane.

1 Introduction

In their celebrated paper [8] Cattaneo and Felder gave a field theoretical interpretation of Kontsevich's formula ([14]) for the deformation quantization of a Poisson manifold (M, Π) where Π stands for the Poisson structure. The field theory derivation involves the so-called Poisson-Sigma model ([13],[16]). This is a two-dimensional topological field theory defined on a surface Σ whose target is a Poisson manifold (M, Π) . The field content is a bundle map from $T\Sigma$ to T^*M . Cattaneo and Felder showed that Kontsevich's formula can be obtained from Feynman expansion of certain Green's functions when Σ is the unit disc D and the base map $X : \Sigma \rightarrow M$ has free boundary conditions.

The same authors proved in [9] that the non-symmetry-breaking boundary conditions of the Poisson-Sigma model are given by coisotropic branes, i.e. submanifolds defined by *first-class constraints*. In this case the quantization of the model is related to the deformation quantization of the coisotropic submanifold.

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We proved recently ([5]) that classically the field X can be consistently restricted at the boundary $\partial\Sigma$ to an almost arbitrary submanifold C . It turns out that the symplectic structure on the reduced phase space of the model is related to the Poisson bracket canonically induced on (a subset of) $C^\infty(C)$.

On the light of these results it is natural to conjecture that the perturbative quantization of the model with general C be related to the deformation quantization of the induced Poisson bracket on (certain functions on) C . In this paper we study in detail the case in which C can be defined by a set of *second-class constraints* which is, in some sense, opposite to the coisotropic one. The quantization of the coisotropic case ([9]) presents some intricacies due to the fact that gauge transformations do not vanish at the boundary. If C is defined by second-class constraints (*second-class brane*) they do vanish and one would expect to have a clean quantization recovering Kontsevich's formula, this time not for Π but for the Dirac bracket on C . We show that this expected result holds and that it emerges in quite a different way from the coisotropic case. Finally, we give the quantization of the Poisson-Sigma model with a general brane defined by a mixture of first and second class constraints.

The paper is organized as follows:

In Section 2 we give a brief introduction to Poisson geometry and Poisson reduction, as well as to the problem of deformation quantization and Kontsevich's solution.

Section 3 recalls the results of [5] on the study of the classically consistent boundary conditions for the Poisson-Sigma model.

Section 4 is devoted to the problem of perturbative quantization of the Poisson-Sigma model with non-coisotropic branes. First, we prove that in the perturbative expansion as defined in [8] and [9] the propagator does not exist when second-class constraints are involved. We show that with slight modifications a well-defined perturbative expansion can be given. However, this first solution is still too naive. In subsection 4.2.2 we take a more original approach and show that the perturbative quantization of the model with target (M, Π) and with a second-class brane gives Kontsevich's formula corresponding to the Dirac bracket on C obtained by reduction from Π .

2 Reduction of Poisson manifolds

This subsection is a brief summary of some results on Poisson reduction presented in [5]. We refer the reader to that paper for details.

Given a Poisson manifold (M, Π) and a closed submanifold $C \hookrightarrow M$, we would like to know whether Π defines in a canonical way a Poisson bracket on $C^\infty(C)$ or at least on a subset of it.

We adopt the notation $\mathcal{A} = C^\infty(M)$ and take the ideal (with respect to the

point-wise product of functions in \mathcal{A})

$$\mathcal{I} = \{f \in \mathcal{A} | f(p) = 0, p \in C\}$$

We view $C^\infty(C)$ as \mathcal{A}/\mathcal{I} .

Define $\mathcal{F} \subset \mathcal{A}$ as the set of *first-class functions*, also called the *normalizer* of \mathcal{I} ,

$$\mathcal{F} = \{f \in \mathcal{A} | \{f, \mathcal{I}\} \subset \mathcal{I}\}.$$

Note that due to the Jacobi identity and the Leibniz rule \mathcal{F} is a Poisson subalgebra of \mathcal{A} and $\mathcal{F} \cap \mathcal{I}$ is a Poisson ideal of \mathcal{F} . Then, we have canonically defined a Poisson bracket in the quotient $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$.

Now, we define the map

$$\begin{aligned} \phi: \mathcal{F}/(\mathcal{F} \cap \mathcal{I}) &\longrightarrow \mathcal{A}/\mathcal{I} \\ f + \mathcal{F} \cap \mathcal{I} &\longmapsto f + \mathcal{I} \end{aligned} \quad (2.1)$$

which is an injective homomorphism of abelian, associative algebras with unit and then induces a Poisson algebra structure $\{.,.\}_C$ on the image, i.e.:

$$\{f_1 + \mathcal{I}, f_2 + \mathcal{I}\}_C = \{f_1, f_2\} + \mathcal{I}. \quad f_1, f_2 \in \mathcal{F}. \quad (2.2)$$

Remark: Note that the elements of $\mathcal{F} \cap \mathcal{I}$ are, in the language of physicists, the generators of *gauge transformations* or, in Dirac's terminology, the *first-class constraints*.

In case ϕ is onto we have endowed C with a canonical Poisson structure. As shown in ref. [5] this situation is equivalent to the existence of what Vaisman defines as an *algebraically Π -compatible normal bundle* of C , see ref. [17] for details. A different strategy for endowing C with a canonical Poisson structure is via the reduction of Dirac structures, see ref. [10]. Here is denoted *Poisson-Dirac* the submanifold for which this reduction actually defines a Poisson structure on C . It is easy to see that if map ϕ is onto, C is Poisson-Dirac and the induced Poisson structure obtained in both ways is the same. We refer the reader to ref. [5] for more details on the relations between the two approaches.

In general, however, ϕ is not onto and C cannot be made a Poisson manifold. What we have is a Poisson bracket on $\phi(\mathcal{F}/(\mathcal{F} \cap \mathcal{I})) \subseteq \mathcal{A}/\mathcal{I}$. The image of ϕ is not easy to characterize in the general case but, as we shall see next, it has a nice interpretation if certain regularity conditions are met.

Let N^*C (or $\text{Ann}(TC)$) be the *conormal bundle* of C (or *annihilator* of TC), i.e. the subbundle of the pull-back $i^*(T^*M)$ consisting of covectors that kill all vectors in TC . Define also the set of *gauge-invariant functions*

$$\mathcal{A}_{inv} := \{f \in \mathcal{A} | \{f, \mathcal{F} \cap \mathcal{I}\} \subset \mathcal{I}\}.$$

We have the following

Theorem:

If $\dim(\Pi_p^\sharp(N_p^*C) + T_pC)$ is constant for every $p \in C$, then $\phi(\mathcal{F}/\mathcal{F} \cap \mathcal{I}) = \mathcal{A}_{inv}/\mathcal{I}$. In other words, the image of ϕ are the gauge-invariant functions restricted to C .

Proof: See [5]. □

The meaning of what we shall call the *strong regularity condition*

$$\dim(\Pi_p^\sharp(N_p^*C) + T_pC) = k + \dim(C), \quad \forall p \in C \quad (2.3)$$

for a non-negative constant k , is clarified by noticing that it allows to choose in a neighborhood $\mathcal{U} \subset M$ of every $p \in C$ adapted local coordinates on M , (X^a, X^μ, X^A) , with $a = 1, \dots, \dim(C)$, $\mu = \dim(C) + 1, \dots, \dim(M) - k$ and $A = \dim(M) - k + 1, \dots, \dim(M)$, verifying:

- (i) $C \cap \mathcal{U}$ is defined by $X^\mu = X^A = 0$.
- (ii) $\{X^\mu, X^\nu\}|_{C \cap \mathcal{U}} = \{X^\mu, X^A\}|_{C \cap \mathcal{U}} = 0$, i.e. X^μ are *first-class constraints*.
- (iii) $\det(\{X^A, X^B\}(p)) \neq 0$, $\forall p \in C \cap \mathcal{U}$, i.e. X^A are *second-class constraints*.

It is clear that in these adapted coordinates the Poisson structure satisfies:

$$\Pi^{\mu\nu}|_{C \cap \mathcal{U}} = 0, \quad \Pi^{A\mu}|_{C \cap \mathcal{U}} = 0, \quad \det(\Pi^{AB})|_{C \cap \mathcal{U}} \neq 0 \quad (2.4)$$

Notice also (Lemma 1 of ref. [5]) that the strong regularity condition is equivalent to

$$N_p^*C \cap \Pi_p^{\sharp-1}(T_pC) = \{(df)_p | f \in \mathcal{F} \cap \mathcal{I}\}, \quad \forall p \in C \quad (2.5)$$

Remark: C is said to be coisotropic if $\Pi^\sharp(N^*C) \subseteq TC$. For such C the strong regularity condition (2.3) is obviously satisfied and every constraint is first-class. In addition, (2.2) is the original bracket on M restricted to the gauge-invariant functions on C . The case of free boundary conditions, $C = M$, is an extreme example of coisotropic submanifold.

Later on we shall be concerned with the situation in which every constraint defining C is second-class, i.e. there are no Greek indices and the strong regularity condition is fulfilled. We call such C a *second-class submanifold* or *second-class brane*. In this case the matrix of the Poisson brackets of the constraints $\Pi^{AB} = \{X^A, X^B\}$ is invertible on C . Defining on C the matrix ω_{AB} by $\omega_{AB}\Pi^{BC} = \delta_A^C$ the Poisson bracket (2.2) can be written locally:

$$\{f + \mathcal{I}, f' + \mathcal{I}\}_C = \{f, f'\} - \{f, X^A\}\omega_{AB}\{X^B, f'\} + \mathcal{I} \quad (2.6)$$

which is the usual definition of the *Dirac bracket* restricted to C . In this case every function on M is trivially gauge-invariant (since $\mathcal{F} \cap \mathcal{I} = 0$), the image of ϕ is $C^\infty(C)$ and we get a Poisson structure on C . In adapted coordinates the components of the canonical Poisson tensor on C corresponding to (2.6) are given by:

$$\Pi_{\mathcal{D}}^{ab} = \Pi^{ab} - \Pi^{aA} \omega_{AB} \Pi^{Bb} \quad (2.7)$$

where the subscript \mathcal{D} stands for Dirac.

When first-class constraints are present one can still use formula (2.7). Given a choice of adapted coordinates $\{X^a, X^\mu, X^A\}$ the expression

$$\Pi_{\mathcal{D}}^{pq} = \Pi^{pq} - \Pi^{pA} \omega_{AB} \Pi^{Bq} \quad (2.8)$$

with the indices $p, q = 1, \dots, \dim(M) - k$ running over a and μ values, defines a Poisson bracket in the submanifold C' on which the second-class constraints vanish (we assume $\det(\Pi^{AB}) \neq 0$ on C'). The submanifold C' is not uniquely defined, as it depends of the concrete choice of the set of second class constraints. C is now a coisotropic submanifold of C' and the Poisson algebra induced by $\Pi_{\mathcal{D}}$ on the gauge invariant functions on C is indeed canonical, independent of the choice of C' , and equals the one given by (2.2).

One can also extend the Poisson tensor to a tubular neighborhood of C' by taking $\Pi_{\mathcal{D}}^{Ap} = \Pi_{\mathcal{D}}^{AB} = 0$. If one considers the tubular neighborhood equipped with the Dirac bracket, C is coisotropic in it and C' is a Poisson submanifold.

For later purposes it is useful to consider the following *weak regularity condition*:

$$\dim\{(df)_p | f \in \mathcal{F} \cap \mathcal{I}\} = k + \dim(M) - \dim(C), \quad \forall p \in C \quad (2.9)$$

for some non-negative constant k .

That the strong regularity condition (2.3) implies the weak one with the same value for the constant k is clear from (2.5).

The weak regularity condition is equivalent to the existence of local coordinates on a tubular neighborhood of every patch of C with a maximal (and constant) number of coordinates which are first-class constraints. In other words, (2.9) holds if and only if there exist local coordinates satisfying (i), (ii) as above and

$$(iii)' \det(\{X^A, X^B\}(p)) \neq 0 \quad \text{for } p \text{ in an open dense subset of } C \cap \mathcal{U}.$$

However, in general the weak regularity condition is not enough to guarantee that $\phi(\mathcal{F}/\mathcal{F} \cap \mathcal{I}) = \mathcal{A}_{inv}/\mathcal{I}$.

3 Poisson-Sigma models on surfaces with boundary

The Poisson-Sigma model is a two-dimensional topological Sigma model defined on a surface Σ and with a finite dimensional Poisson manifold (M, Π) as target. The fields of the model are given by a bundle map $(X, \eta) : T\Sigma \rightarrow T^*M$ consisting of a base map $X : \Sigma \rightarrow M$ and a 1-form η on Σ with values in the pullback by X of the cotangent bundle of M . The action functional has the form¹

$$S(X, \eta) = \int_{\Sigma} \langle \eta, \wedge dX \rangle + \frac{1}{2} \langle \Pi \circ X, \eta \wedge \eta \rangle, \quad (3.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between vectors and covectors of M .

If X^i are local coordinates in M , σ^κ , $\kappa = 1, 2$ local coordinates in Σ , Π^{ij} the components of the Poisson structure in these coordinates and $\eta_i = \eta_{i\kappa} d\sigma^\kappa$, the action reads

$$S(X, \eta) = \int_{\Sigma} \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j \quad (3.2)$$

The equations of motion in the bulk are:

$$dX^i + \Pi^{ij}(X) \eta_j = 0 \quad (3.3a)$$

$$d\eta_i + \frac{1}{2} \partial_i \Pi^{jk}(X) \eta_j \wedge \eta_k = 0 \quad (3.3b)$$

The infinitesimal transformations

$$\delta_\epsilon X^i = \Pi^{ij}(X) \epsilon_j \quad (3.4a)$$

$$\delta_\epsilon \eta_i = -d\epsilon_i - \partial_i \Pi^{jk}(X) \eta_j \epsilon_k \quad (3.4b)$$

where $\epsilon = \epsilon_i dX^i$ is a section of $X^*(T^*M)$, change the action (3.2) by a boundary term

$$\delta_\epsilon S = - \int_{\Sigma} d(dX^i \epsilon_i). \quad (3.5)$$

Notice that

$$[\delta_\epsilon, \delta_{\epsilon'}] X^i = \delta_{[\epsilon, \epsilon']^*} X^i \quad (3.6a)$$

$$[\delta_\epsilon, \delta_{\epsilon'}] \eta_i = \delta_{[\epsilon, \epsilon']^*} \eta_i - \epsilon_k \epsilon'_l \partial_i \partial_j \Pi^{kl} (dX^j + \Pi^{js}(X) \eta_s) \quad (3.6b)$$

where $[\epsilon, \epsilon']^*_k := -\partial_k \Pi^{ij}(X) \epsilon_i \epsilon'_j$. The term in parenthesis in (3.6b) is the equation of motion (3.3a). Hence, the commutator of two transformations of type (3.4) is

¹We adopt throughout this paper the notation and sign conventions of Cattaneo and Felder's paper [8].

a transformation of the same type only on-shell and the gauge transformations (3.4) form an open-algebra.

If Σ has a boundary a new term appears in the variation of the action under a change of X when performing the integration by parts:

$$\delta_X S = - \int_{\partial\Sigma} \delta X^i \eta_i + \int_{\Sigma} \delta X^i (d\eta_i + \frac{1}{2} \partial_i \Pi^{jk}(X) \eta_j \wedge \eta_k) \quad (3.7)$$

Let us restrict the field X at the boundary to a closed submanifold C of M :

$$X|_{\partial\Sigma} : \partial\Sigma \rightarrow C \subset M \quad (3.8)$$

The conditions for η should make the boundary term in (3.7) vanish and make the equations of motion (3.3a) consistent at the boundary. This is achieved if we take the following boundary conditions (BC) for the fields:

$$(\text{XBC}) \quad X(m) \in C, \forall m \in \partial\Sigma$$

$$(\eta\text{BC}) \quad \eta_{\mathbf{t}}(m) \in \{(df)_m | f \in \mathcal{F} \cap \mathcal{I}\} \forall m \in \partial\Sigma$$

where $\eta_{\mathbf{t}} = \eta_{it} dX^i$ is the contraction of η with vector fields tangent to the boundary. In order to have gauge transformations compatible at the boundary with the BC one must have

$$(\epsilon\text{BC}) \quad \epsilon(m) \in \{(df)_m | f \in \mathcal{F} \cap \mathcal{I}\} \forall m \in \partial\Sigma$$

Clearly, (XBC) is preserved by (3.4a). In reference [5] it is proven that if the strong regularity condition (2.3) holds, and in the adapted coordinates described in section 2.1, the gauge transformation (3.4b) also preserves (ηBC). Here we redo the proof assuming only the **weak regularity condition**.

We take the adapted coordinates of section 2.1 $\{X^a, X^\mu, X^A\}$ where we use a, b, \dots for the coordinates on C , μ, ν, \dots for the first-class constraints and A, B, \dots for the second-class ones. In these coordinates $\{df | f \in \mathcal{F} \cap \mathcal{I}\}$ is spanned locally by dX^μ and the BC read:

$$(\text{XBC}) \quad X^\mu(m) = X^A(m) = 0, \forall m \in \partial\Sigma$$

$$(\eta\text{BC}) \quad \eta_{at}(m) = \eta_{At}(m) = 0, \forall m \in \partial\Sigma$$

$$(\epsilon\text{BC}) \quad \epsilon_a(m) = \epsilon_A(m) = 0, \forall m \in \partial\Sigma.$$

The consistency of these BC require that gauge variations of η_{at} and η_{At} vanish at the boundary. For first-class components

$$\delta_\epsilon \eta_{at}(m) = -\partial_a \Pi^{\mu\nu}(X(m)) \eta_{\mu\mathbf{t}}(m) \epsilon_\nu(m) = 0, \quad \forall m \in \partial\Sigma$$

because ∂_a is a derivative in a direction tangent to C and $\Pi^{\mu\nu}$ vanishes on C .

In order to prove the vanishing at the boundary of the transformation of *second-class* components

$$\delta_\epsilon \eta_{At}(m) = -\partial_A \Pi^{\mu\nu}(X(m)) \eta_{\mu t}(m) \epsilon_\nu(m) = 0, \quad m \in \partial\Sigma \quad (3.9)$$

one needs the following Jacobi identity

$$\begin{aligned} & \Pi^{AB} \partial_A \Pi^{\mu\nu} + \Pi^{\gamma B} \partial_\gamma \Pi^{\mu\nu} + \Pi^{aB} \partial_a \Pi^{\mu\nu} \\ & + \Pi^{A\nu} \partial_A \Pi^{B\mu} + \Pi^{\gamma\nu} \partial_\gamma \Pi^{B\mu} + \Pi^{a\nu} \partial_a \Pi^{B\mu} \\ & + \Pi^{A\mu} \partial_A \Pi^{\nu B} + \Pi^{\gamma\mu} \partial_\gamma \Pi^{\nu B} + \Pi^{a\mu} \partial_a \Pi^{\nu B} = 0 \end{aligned} \quad (3.10)$$

Evaluating the previous expression on C and using $\Pi^{\mu\nu}|_C = \Pi^{\mu A}|_C = 0$ one has

$$\Pi^{AB} \partial_A \Pi^{\mu\nu}|_C = 0.$$

From the fact that $\det(\Pi^{AB}) \neq 0$ in an open dense subset of C as implied by the weak regularity condition, an argument of continuity shows that $\partial_A \Pi^{\mu\nu}|_C = 0$ and $\delta_\epsilon \eta_{At}$ vanishes at the boundary of Σ .

We shall call *weakly regular brane* to a submanifold C which satisfies the weak regularity condition. As shown above, weakly regular branes lead to consistent boundary conditions for the Poisson Sigma model at the classical level.

4 Quantization of the Poisson-Sigma model

4.1 Batalin-Vilkovisky procedure

Here we recall the steps followed in [8] to construct the partition function of the Poisson-Sigma model².

Consider the space of fields³ with a \mathbb{Z} gradation corresponding to the ghost number and a \mathbb{Z}_2 gradation corresponding to the Grassmann parity. The standard BRST formalism for the quantization of a theory with gauge symmetries introduces anticommuting scalar fields β_i (ghosts) and γ^i (antighosts) along with commuting scalar fields λ^i (Lagrange multipliers). The basic ghost number assignments are: $\text{gh}(X^i) = \text{gh}(\eta_i) = \text{gh}(\lambda^i) = 0$, $\text{gh}(\beta_i) = 1$, $\text{gh}(\gamma^i) = -1$.

²The path integral quantization of the Poisson-Sigma model in the particular case of 2D gravity was first carried out in [15].

³In this subsection we are not concerned with the boundary conditions of the additional fields entering the formalism. They are discussed in detail in the next subsection.

Now one defines an odd derivation of ghost number one δ_0 :

$$\delta_0 X^i = \Pi^{ij}(X)\beta_j \quad (4.1a)$$

$$\delta_0 \eta_i = -d\beta_i - \partial_i \Pi^{jk}(X)\eta_j \beta_k \quad (4.1b)$$

$$\delta_0 \beta_i = \frac{1}{2} \partial_i \Pi^{jk}(X)\beta_j \beta_k \quad (4.1c)$$

$$\delta_0 \gamma^i = \lambda^i \quad (4.1d)$$

$$\delta_0 \lambda^i = 0 \quad (4.1e)$$

extended to functions of the fields through the Leibniz rule. The problem is that since the gauge transformations (3.4) close only on-shell, δ_0^2 vanishes only on-shell and we do not have a well-defined cohomology on the space of fields.

The extension of the BRST scheme which works for open algebras is known as Batalin-Vilkovisky (BV) procedure⁴. Firstly, we double the number of fields by introducing an antifield φ_i^+ for each field φ^i (φ stands here for X, η, β, γ and λ) such that φ_i^+ has Grassmann parity opposite to that of φ^i and $\text{gh}(\varphi_i^+) = -1 - \text{gh}(\varphi^i)$.

The partition function is given by

$$Z = \int e^{\frac{i}{\hbar} S_{BV}} \delta \left(\varphi_i^+ - \frac{\overrightarrow{\delta} \Psi}{\delta \varphi^i} \right) \mathcal{D}\varphi \mathcal{D}\varphi^+ \quad (4.2)$$

where

$$\begin{aligned} S_{BV} = & \int_D \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j + X_i^+ \Pi^{ij}(X) \beta_j - \\ & - \eta^{+i} \wedge (d\beta_i + \partial_i \Pi^{kl}(X) \eta_k \beta_l) - \frac{1}{2} \beta^{+i} \partial_i \Pi^{jk}(X) \beta_j \beta_k - \\ & - \frac{1}{4} \eta^{+i} \wedge \eta^{+j} \partial_i \partial_j \Pi^{kl}(X) \beta_k \beta_l - \lambda^i \gamma_i^+ \end{aligned} \quad (4.3)$$

and the *gauge-fixing fermion* Ψ is an anticommuting functional of the fields of ghost number -1 which makes the path integral well-defined. The canonical choice is to take Ψ as the scalar product of the antighosts and the gauge-fixing conditions.

4.2 Quantization on the disc

Take Σ the unit disc $D = \{\sigma \in \mathbb{R}^2, |\sigma| \leq 1\}$. Cattaneo and Felder showed in [8] that the perturbative expansion of certain Green's functions of the Poisson-Sigma model defined on D with $C = M$ (free BC) yields Kontsevich's \star -product corresponding to the Poisson manifold M , namely

$$\langle f(X(0))g(X(1))\delta(X(\infty) - x) \rangle = f \star g(x) \quad (4.4)$$

⁴See the original papers [1],[2] and the excellent exposition included in [11].

where 0, 1 and ∞ are three cyclically ordered points at the boundary of D and the expectation value is calculated using (4.2). In a later work ([9]) the same authors studied the quantization with a general coisotropic brane C , which turned out to be related with the deformation quantization of the submanifold $C \hookrightarrow M$.

In both papers [8],[9] the Green's functions (4.4) are worked out in the same fashion: first, one takes the *Lorentz gauge* $d*\eta = 0$, where the Hodge operator acting on 1-forms requires the introduction of a complex structure on D . The Feynman expansion in powers of \hbar is then performed around the constant classical solution where $X(\sigma) = x \in C$ and the rest of the fields vanish. In this case expanding in powers of \hbar amounts to expanding in powers of Π or, equivalently, to expanding around zero Poisson structure.

In the next subsection we shall try to work out (4.4) following the steps enumerated in the last paragraph when C is non-coisotropic. We shall see that when second-class constraints are present the propagator does not exist, but a natural redefinition of the unperturbed (or quadratic) part of the action will yield a well-defined perturbative expansion, showing that the non-coisotropic branes also make sense at the quantum level. However, this leads to a messy expression whose interpretation is far from clear. In the subsection 4.2.2 we shall see that a change of the gauge fixing is illuminating in order to unravel the relation between the quantization of the Poisson-Sigma model with a non-coisotropic brane and Kontsevich's formula.

4.2.1 The perturbation expansion in the non-coisotropic case

Let us take the *Lorentz gauge* $d*\eta = 0$ as said above. The gauge fixing fermion is then:

$$\Psi = \int_D \gamma^i d*\eta_i$$

and the gauge fixed action with the antifields integrated out is

$$\begin{aligned} S_{gf} &= \int_D \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - *d\gamma^i \wedge (d\beta_i + \partial_i \Pi^{kl}(X) \eta_k \beta_l) - \\ &- \frac{1}{4} *d\gamma^i \wedge *d\gamma^j \wedge \partial_i \partial_j \Pi^{kl}(X) \beta_k \beta_l - \lambda^i d*\eta_i \end{aligned} \quad (4.5)$$

Now write $X^i(\sigma) = x^i + \xi^i(\sigma)$ and choose

$$S_0 = \int_D \eta_i \wedge d\xi^i - *d\gamma^i \wedge d\beta_i - d*\eta_i \lambda^i \quad (4.6)$$

as the quadratic part which defines the propagators whereas $S_{pert} = S_{gf} - S_0$ yields the vertices of the perturbative expansion (see [8] for explicit expressions). Our aim is to show that for non-coisotropic C the propagator cannot fulfill the appropriate BC.

In the adapted coordinates of section 2.1 index i splits into a, μ, A where ξ^a are coordinates along the brane (free at the boundary) and ξ^μ and ξ^A are respectively first-class and second-class coordinates transversal to the brane and must vanish at the boundary. For the rest of the fields we have Dirichlet boundary conditions for $\lambda^a, \lambda^A, \eta_{at}, (*\eta_\mu)_t, \eta_{At}, \beta_a, \beta_A, \gamma^a$ and γ^A and Neumann boundary conditions for β_μ, γ^μ and λ^μ .

It is convenient to map conformally the disc onto the upper complex half plane H_+ (recall the conformal invariance of S_{gf}) and use a complex coordinate $z \in H_+$. The propagators for the β_i and γ^i fields are given by the Green's function of the Laplacian with Dirichlet boundary conditions for components along the brane and for second-class constraints and Neumann boundary conditions for first-class constraints, i.e.

$$\begin{aligned}\langle \gamma^a(w, \bar{w}) \beta_b(z, \bar{z}) \rangle_0 &= \frac{i\hbar}{2\pi} \delta_b^a \log \frac{|w - z|}{|w - \bar{z}|} \\ \langle \gamma^\mu(w, \bar{w}) \beta_\nu(z, \bar{z}) \rangle_0 &= \frac{i\hbar}{2\pi} \delta_\nu^\mu \log(|w - z||w - \bar{z}|) \\ \langle \gamma^A(w, \bar{w}) \beta_B(z, \bar{z}) \rangle_0 &= \frac{i\hbar}{2\pi} \delta_B^A \log \frac{|w - z|}{|w - \bar{z}|}\end{aligned}\tag{4.7}$$

The other non vanishing components of the propagator are conveniently expressed in terms of the complex fields $\zeta^j = \xi^j + i\lambda^j$ and $\bar{\zeta}^j = \xi^j - i\lambda^j$.

The general solution of the corresponding Schwinger-Dyson equations for the unperturbed action is:

$$\begin{aligned}\langle \zeta^i(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 &= \frac{\hbar}{2\pi} \delta_j^i \left(-\frac{dz}{w - z} + f_j(w, z)dz + g_j(w, \bar{z})d\bar{z} \right) \\ \langle \bar{\zeta}^i(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 &= \frac{\hbar}{2\pi} \delta_j^i \left(\frac{d\bar{z}}{\bar{w} - \bar{z}} + \bar{f}_j(\bar{w}, \bar{z})d\bar{z} + \bar{g}_j(\bar{w}, z)dz \right)\end{aligned}\tag{4.8}$$

where no sum in j is assumed and $f_j, \bar{f}_j, g_j, \bar{g}_j$ are holomorphic in their arguments with domains given by $w, z \in H_+$. The boundary conditions imply $f_a = f_\mu = \bar{f}_a = \bar{f}_\mu = 0$ and

$$g_a(w, \bar{z}) = -g_\mu(w, \bar{z}) = \frac{1}{w - \bar{z}}, \quad \bar{g}_a(\bar{w}, z) = -\bar{g}_\mu(\bar{w}, z) = \frac{-1}{\bar{w} - z}$$

However, if we try to fulfill the boundary conditions for the components corresponding to the second-class constraints (A, B, \dots indices) we find a contradiction as $f_A(w, z)$ and $\bar{f}_A(w, z)$ must extend to entire functions in w and z and besides

$$\bar{f}_A(w, \bar{z}) - f_A(w, \bar{z}) = -\frac{2}{w - \bar{z}}$$

which is obviously impossible and the propagator does not exist.

At this point one might be tempted to conclude that only the coisotropic branes make sense at the quantum level. But this is a too sloppy conclusion since we have only shown that the perturbative expansion defined by the choice of (4.6) as the unperturbed part ceases to exist when second-class constraints appear. The question is whether there is a different definition of the perturbative expansion leading to a well-defined result in this case. The situation is reminiscent to the contradiction found by Dirac in imposing the second-class constraints on the states of the physical Hilbert space; he proposed to circumvent this difficulty with the help of the Dirac bracket [12].

From now on we shall restrict to branes which satisfy the **strong regularity condition** (2.3) and, in order to make the presentation simpler, we shall assume that there are no first-class constraints, i.e. we restrict to the **second-class branes** of section 2.1.

The strategy for solving the problem is the well-known technique of using our opponent's strength against him. The origin of the non-existence of the propagator for the second-class coordinates is that $\det(\Pi^{AB}) \neq 0$ implies that if $X^A = 0$ at the boundary then η_{At} must also vanish. And the propagator cannot satisfy these two conditions simultaneously. But precisely due to the fact that $\det(\Pi^{AB}) \neq 0$ and given that the η_A fields appear at most quadratically in the gauge fixed action (4.5) we can perform the Gaussian integration over them in order to get an effective action S^{eff} . This action can be used to compute the correlation functions of observables that do not involve η_A fields as it is our case.

Once the integration has been performed there is a splitting of S^{eff} which defines a consistent perturbative expansion. Take $S^{eff} = S_0^{eff} + S_{pert}^{eff}$ with

$$S_0^{eff} = \int_D \eta_a \wedge d\xi^a - d*\eta_a \lambda^a + \omega_{AB}(x) d\xi^A \wedge *d\lambda^B - *\gamma^i \wedge d\beta_i \quad (4.9)$$

The β, γ propagators are as before (see eq. (4.7)), as well as those for ζ^a and η_a . In addition, S_0^{eff} yields well-defined propagators for the other fields, the only non-zero components being

$$\langle \lambda^A(w, \bar{w}) \xi^B(z, \bar{z}) \rangle_0^{eff} = \frac{i\hbar}{2\pi} \Pi^{AB}(x) \log \frac{|w - z|}{|w - \bar{z}|}. \quad (4.10)$$

Now one can expand S_{pert}^{eff} into vertices and define a perturbative expansion for Green's functions of the form (4.4). However, from the resulting perturbative series it seems very hard to find out whether the formula (4.4) defines an associative product. A simpler derivation that gives a positive answer is given in the next subsection.

4.2.2 Second-class branes, Kontsevich's formula and Dirac bracket

Let us take advantage of our opponent's strength in a more profound sense. Using that Π^{AB} is invertible in every point of C and consequently in a tubular neighborhood of C we can show that the gauge fixing

$$d*\eta_a = 0, \quad \xi^A = 0 \quad (4.11)$$

is reachable, at least locally:

$d*\eta_a = 0$ can be obtained by choosing suitably ϵ_a in (3.4b). Now, write (3.4a) for upper-case Latin indices

$$\delta_\epsilon \xi^A = \Pi^{Aa}(x + \xi)\epsilon_a + \Pi^{AB}(x + \xi)\epsilon_B.$$

Since Π^{AB} is invertible one can solve for ϵ_B and get $\xi^A = 0$.

We want to stress that the analog of (4.11) is not an admissible gauge-fixing in the coisotropic case. For second-class branes both the Lorentz gauge and (4.11) are admissible but, as we shall see, the latter makes the perturbative quantization transparent and is the appropriate approach to the problem.

Let us go back to the BV action (4.3), set the indices of the antighosts γ and Lagrange multipliers λ upstairs or downstairs as demanded by (4.11) and take

$$\Psi = \int_D \gamma^a d*\eta_a + \int_D \gamma_A X^A \quad (4.12)$$

where γ_A are anticommuting 2-form fields on D .

On the submanifold $\varphi_i^+ = \frac{\vec{\delta}\Psi}{\delta\varphi^i}$ we have

$$\begin{aligned} \beta^{+a} &= \beta^{+A} = 0 \\ \eta^{+a} &= *d\gamma^a, \eta^{+A} = 0 \\ X_a^+ &= 0, X_A^+ = -\gamma_A \\ \gamma_a^+ &= d*\eta_a, \gamma^{+A} = X^A \end{aligned}$$

And the gauge fixed action with the antifields integrated out reads now

$$\begin{aligned} \tilde{S}_{gf} &= \int_D \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - *d\gamma^a \wedge (d\beta_a + \partial_a \Pi^{kl}(X) \eta_k \beta_l) - \\ &- \frac{1}{4} *d\gamma^a \wedge *d\gamma^b \partial_a \partial_b \Pi^{kl}(X) \beta_k \beta_l - \lambda^a d*\eta_a - \gamma_A \Pi^{Ai}(X) \beta_i - \lambda_A X^A \end{aligned}$$

Recall that we are interested in calculating the expectation value of functionals depending only on X . Hence, integration over λ_A sets $X^A = 0$ and we can write:

$$\begin{aligned} \tilde{S}'_{gf} &= \int_D \eta_a \wedge dX^a + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - *d\gamma^a \wedge (d\beta_a + \partial_a \Pi^{kl}(X) \eta_k \beta_l) - \\ &- \frac{1}{4} *d\gamma^a \wedge *d\gamma^b \partial_a \partial_b \Pi^{kl}(X) \beta_k \beta_l - \lambda^a d*\eta_a - *\gamma_A \Pi^{Ai}(X) \beta_i \end{aligned}$$

where Π is evaluated on $X^A = 0$.

Now, integrating over γ_A forces

$$\Pi^{Ai}\beta_i = 0 \Leftrightarrow \beta_A = -\omega_{AB}\Pi^{Ba}\beta_a \quad (4.13)$$

which is a crucial relation which can be used to get rid of the components of the fields with upper-case indices and get an effective action depending only on the lower-case components. Notice that writing

$$*\mathrm{d}\gamma^a \wedge \partial_a \Pi^{kl}(X)\eta_k\beta_l = *\mathrm{d}\gamma^a \wedge \partial_a (\Pi^{bc}(X)\eta_b\beta_c + \Pi^{Ai}(X)\eta_A\beta_i + \Pi^{bA}(X)\eta_b\beta_A)$$

and applying (4.13) to the second and third terms in parentheses we obtain the Dirac Poisson structure (2.7) in a beautiful way:

$$*\mathrm{d}\gamma^a \wedge \partial_a \Pi^{kl}\eta_k\beta_l = *\mathrm{d}\gamma^a \wedge \partial_a \Pi_{\mathcal{D}}^{bc}\eta_b\beta_c$$

Doing the same for the term quadratic in $\mathrm{d}\gamma$ we get:

$$\begin{aligned} \tilde{S}_{gf}'' &= \int_D \eta_a \wedge \mathrm{d}X^a + \frac{1}{2}\Pi^{ij}(X)\eta_i \wedge \eta_j - *\mathrm{d}\gamma^a \wedge (\mathrm{d}\beta_a + \partial_a \Pi_{\mathcal{D}}^{bc}(X)\eta_b\beta_c) - \\ &- \frac{1}{4}*\mathrm{d}\gamma^a \wedge *\mathrm{d}\gamma^b \partial_a \partial_b \Pi_{\mathcal{D}}^{cd}(X)\beta_c\beta_d - \lambda^a \mathrm{d}*\eta_a - i\hbar \log \det(\Pi^{AB}(X)) \end{aligned}$$

where the last term in the action comes from the Jacobian corresponding to the delta distribution

$$\delta(\Pi^{Ai}\beta_i) = \delta(\beta_A + \omega_{AB}\Pi^{Ba}\beta_a) \det(\Pi^{BC}(X))$$

The final step is to integrate out the η_A fields. The integral is Gaussian (due again to the non-degeneracy of Π^{AB}) and the determinant coming from it cancels the contribution from the δ function. Finally,

$$\begin{aligned} \tilde{S}_{gf}^{eff} &= \int_D \eta_a \wedge \mathrm{d}X^a + \frac{1}{2}\Pi_{\mathcal{D}}^{ab}(X)\eta_a \wedge \eta_b - *\mathrm{d}\gamma^a \wedge (\mathrm{d}\beta_a + \partial_a \Pi_{\mathcal{D}}^{cd}(X)\eta_c\beta_d) - \\ &- \frac{1}{4}*\mathrm{d}\gamma^a \wedge *\mathrm{d}\gamma^b \partial_a \partial_b \Pi_{\mathcal{D}}^{cd}(X)\beta_c\beta_d - \lambda^a \mathrm{d}*\eta_a \end{aligned} \quad (4.14)$$

which is Cattaneo and Felder's gauge-fixed BV action for a Poisson-Sigma model defined on D , with target $(C, \Pi_{\mathcal{D}})$ (recall that we set $X^A = 0$) and boundary conditions such that X^a is free and η_a vanishes on vectors tangent to ∂D . In other words, we have ended up with the situation studied in [8]. Invoking the results therein we can deduce our announced relation, namely that the perturbative expansion of

$$\langle f(X(0))g(X(1))\delta(X(\infty) - x) \rangle, \quad f, g \in C^\infty(M)$$

yields Kontsevich's formula for $\Pi_{\mathcal{D}}$ applied to the restrictions to C of f and g .

In the derivation of this result the second-class boundary conditions seem to play no role, as they are not used to compute any propagator. Notice, however, that the gauge fixing (4.11) makes sense only if the fields ξ^A vanish at the boundary before fixing the gauge. As stressed above the fact that $\det(\Pi^{AB}) \neq 0$ is also essential. This ties inextricably the present result and the use of second-class branes together.

We would like to stress the interesting cancellation of the determinant coming from the integration of the γ_A and β_A fields with that coming from the integration of the η_A fields. It would be worth finding out whether there is some underlying symmetry behind it.

4.2.3 Quantization with a general strongly regular brane

Once the quantization of the Poisson-Sigma model with a second-class brane has been understood, it is straightforward to describe the procedure for the quantization of the model with an arbitrary *strongly regular brane*⁵ defined by both first and second-class constraints. The appropriate gauge fixing fermion in the general case is

$$\Psi = \int_D \gamma^a d*\eta_a + \gamma^\mu d*\eta_\mu + \gamma_A X^A \quad (4.15)$$

Then, we integrate out the second-class components of the fields exactly as above and we are left with

$$\begin{aligned} \tilde{S}_{gf}^{eff} &= \int_D \eta_p \wedge dX^p + \frac{1}{2} \Pi_D^{pq}(X) \eta_p \wedge \eta_q - *d\gamma^p \wedge (d\beta_p + \partial_p \Pi_D^{qr}(X) \eta_q \beta_r) - \\ &- \frac{1}{4} *d\gamma^p \wedge *d\gamma^q \partial_p \partial_q \Pi_D^{rs}(X) \beta_r \beta_s - \lambda^p d*\eta_p \end{aligned} \quad (4.16)$$

where the indices now run over a and μ values and $\Pi_{\mathcal{D}}$ is the Dirac bracket of (2.8). This is Cattaneo and Felder's gauge-fixed BV action for the Poisson-Sigma model with target given by local coordinates (X^a, X^μ) , Poisson structure $\Pi_{\mathcal{D}}$ and a coisotropic brane defined by $X^\mu = 0$. At this point we can apply the results of [9].

An interesting question is how the choice of a set of second-class constraints affects the final result. In our case the choice of second class constraints was made through the gauge fixing fermion, i.e. a different choice amounts to a change of gauge-fixing. Since the expectation values of gauge-invariant observables do not depend on this particular choice, we conclude that the final result is independent of the choice of second-class constraints.

⁵A brane satisfying the strong regularity condition.

The whole derivation parallels that of Dirac's quantization of constrained systems ([12]): one gets rid of the second-class constraints by defining the appropriate Dirac bracket that can be quantized with the first class constraints imposed on the states. It is nice that this result is obtained in a quantum-field theoretical context by tuning the boundary conditions of the fields.

Remark: The need for strong regularity in the quantum case can be seen, for example, from the fact that we need $\det(\Pi^{AB}) \neq 0$ in every point of a tubular neighborhood of C in order to perform the Gaussian integration over the η_A fields.

5 Conclusions and further work

We have proven that non-coisotropic branes in the Poisson-Sigma model are allowed not only classically but also in the quantum setup. However, the quantization of the model when second-class constraints are present requires a procedure which differs from the coisotropic case. In particular, the perturbative expansion must be redefined, as pointed out in [4]. After such redefinition and carrying out the calculations with a suitable gauge-fixing we can show that the perturbative quantization of the model on the disc with a second-class brane gives Kontsevich's formula for the Dirac bracket induced on the brane. In a sense these branes are much simpler than the coisotropic ones as its quantization always defines an associative star product and the deformed algebra is defined for all functions restricted to the brane (no "quantum" gauge invariance is required).

Our final result is an expansion in powers of the Dirac Poisson tensor $\Pi_{\mathcal{D}}$. This fact gives an explanation for the non-existence of propagator in the perturbative expansion around zero Poisson structure, which is the one used in the coisotropic case. The formula (2.7) shows that the inverse of Π^{AB} enters in the expression of $\Pi_{\mathcal{D}}$ and therefore it is not perturbatively connected to $\Pi = 0$.

It is interesting how the boundary conditions of the model have such a strong influence in the perturbative expansion. It is somehow similar to the instanton calculations in which one expands around different classical solutions. Here the (fixed) branes play the role of instantons determining the perturbative expansion.

In this paper we studied both classical and quantum branes. In the classical scenario we saw that *weakly regular branes* were allowed generalizing slightly the results of [5]. In this case the phase space of the theory is not a Poisson manifold since it has special points (defects) in which the Poisson structure does not exist. The classical theory is then more efficiently described in terms of a Poisson algebra. Quantization of these branes, which is not available at the moment, might correspond to the deformation quantization of a Poisson algebra (rather than a Poisson manifold), an issue that has some interest on its own. It would be worth applying these results to the case in which the target manifold is a Poisson-Lie group and trying to extend the bulk-boundary duality found in

[6] to more general boundary conditions.

Another interesting topic is that of the relative positions of different branes. In particular one might deform a brane and study how the Green's functions change. In this way one could try to obtain the coisotropic brane as a (singular) limit of the second-class one, which might help to understand the obstructions to associativity and other technical details found in [9].

This paper gives a mechanism to carry out the quantization of second-class submanifolds. The procedure is somehow indirect and uses the Poisson-Sigma model as the key ingredient. In the coisotropic case the reduction has been performed directly at the algebraic level ([3] [7]), providing a quantum counterpart of the classical Poisson reduction. It would be very interesting to investigate this quantum reduction also for non-coisotropic submanifolds.

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